

# The Local Orthogonality between Quantum States and Entanglement Decomposition

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## Abstract

In the paper, we show that when a quantum state can be decomposed as a convex combination of locally orthogonal mixed states, its entanglement can be decomposed into the entanglement of these mixed states without losing them. The obtained result generalizes a corresponding one proved by Horodecki [Acta Phys. Slov. 48, 141 (1998)]. But, for the entanglement cost it requires certain conditions for holding the decomposition, and the distillable entanglement only has a weak result as inequality. Finally, we presented an example to show that the conditions of our conclusions are existence.

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## 1 Introduction and preliminaries

In this paper, we always assume that  $\mathcal{H}_A$ ,  $\mathcal{H}_B$ ,  $\mathcal{K}_A$  and  $\mathcal{K}_B$  are finite dimensional complex Hilbert spaces. Let  $L(\mathcal{H}_A, \mathcal{K}_A)$  be the set of all linear operators from  $\mathcal{H}_A$  to  $\mathcal{K}_A$ . A quantum state  $\rho$  of some quantum system, described by  $\mathcal{H}_A$ , is a positive semi-definite operator of trace one, in particular, for each unit vector  $|\psi\rangle \in \mathcal{H}_A$ , the operator  $\rho = |\psi\rangle\langle\psi|$  is said to be a *pure state*. We

can identify the pure state  $|\psi\rangle\langle\psi|$  with the unit vector  $|\psi\rangle$ . The set of all quantum states on  $\mathcal{H}_A$  is denoted by  $D(\mathcal{H}_A)$ .

For each quantum state  $\rho \in D(\mathcal{H}_A)$ , its *von Neumann entropy* is defined by

$$S(\rho) = -\text{Tr} \rho \log_2 \rho.$$

Let  $p = (p_a) \in \mathbb{R}^\Sigma$  be a probability distribution, the *Shannon entropy*  $H(p)$  of  $p$  is defined by

$$H(p) = -\sum_{a \in \Sigma} p_a \log_2 p_a.$$

For given probability distribution  $p = (p_a) \in \mathbb{R}^\Sigma$ , positive integer number  $n$  and  $\varepsilon > 0$ , we say that a string  $a_1 \cdots a_n \in \Sigma^n = \Sigma \times \Sigma \cdots \Sigma$  is  $\varepsilon$ -typical if

$$2^{-n(H(p)+\varepsilon)} < p_{a_1 \cdots a_n} < 2^{-n(H(p)-\varepsilon)},$$

where  $p_{a_1 \cdots a_n} = p_{a_1} \cdots p_{a_n}$ . The set of all  $\varepsilon$ -typical strings is denoted by  $T_{n,\varepsilon}$ , that is,

$$T_{n,\varepsilon} = \{a_1 \cdots a_n \in \Sigma^n : 2^{-n(H(p)+\varepsilon)} < p_{a_1 \cdots a_n} < 2^{-n(H(p)-\varepsilon)}\}.$$

The  $\varepsilon$ -typical string set  $T_{n,\varepsilon}$  has the following property [1]:

$$\lim_{n \rightarrow \infty} \sum_{a_1 \cdots a_n \in T_{n,\varepsilon}} p_{a_1 \cdots a_n} = 1.$$

For each  $t = a_1 \cdots a_n \in \Sigma^n$ , we denote  $n_{a,t}$  is the times of  $a$  appearing in  $t$ . If  $a$  does not appear in  $t$ , we denote  $n_{a,t} = 0$ . It is clear that  $\sum_{a \in \Sigma} n_{a,t} = n$ . Moreover, we say that a set  $T_\varepsilon^n$  is  $\varepsilon$ -strong typical set [2], if

$$T_\varepsilon^n = \{s \in \Sigma^n : p_s = \prod_{a \in \Sigma} p_a^{n_{a,s}}, n_{a,s} \in [p_a n - \frac{\varepsilon n \log_{p_a} 2}{|\Sigma|}, p_a n + \frac{\varepsilon n \log_{p_a} 2}{|\Sigma|}]\}.$$

We can easily see that  $T_\varepsilon^n \subseteq T_{n,\varepsilon}$ , and similar to  $\varepsilon$ -typical set we have the following property [2], for complete sake, we prove them.

**Lemma 1.1.** Let  $p = (p_a) \in \mathbb{R}^\Sigma$  be a probability distribution and let  $\varepsilon > 0$ , then

$$\lim_{n \rightarrow \infty} \sum_{s \in T_\varepsilon^n} p_s = 1.$$

*Proof.* Let  $X_a^n, X_b^n, \cdots$  be independent for each  $a, b, \cdots \in \Sigma$  and positive integer  $n$ . The random variables are defined as follows: for each  $a \in \Sigma$  randomly according to the probability distribution  $p$ , let  $X_a^n$  be the times of  $p_a$  appears in  $p_{a_1 \cdots a_n}$ . It holds that

$$\frac{X_a^n - np_a}{\sqrt{np_a(1-p_a)}} \sim N(0,1) \text{ (approximate),}$$

when  $n \rightarrow \infty$ .

Therefore, for  $\varepsilon > 0$ ,

$$\begin{aligned} & \Pr[p_a n - \frac{\varepsilon n \log_{p_a} 2}{|\Sigma|} \leq X_a^n \leq p_a n + \frac{\varepsilon n \log_{p_a} 2}{|\Sigma|}] \\ = & \Pr[-\frac{\varepsilon n \log_{p_a} 2}{\sqrt{np_a(1-p_a)}|\Sigma|} \leq \frac{X_a^n - np_a}{\sqrt{np_a(1-p_a)}} \leq \frac{\varepsilon n \log_{p_a} 2}{\sqrt{np_a(1-p_a)}|\Sigma|}], \end{aligned}$$

and  $\frac{\varepsilon n \log_{p_a} 2}{\sqrt{np_a(1-p_a)}|\Sigma|} \rightarrow \infty$  as  $n \rightarrow \infty$ , so we have

$$\lim_{n \rightarrow \infty} \Pr[p_a n - \frac{\varepsilon n \log_{p_a} 2}{|\Sigma|} \leq X_a^n \leq p_a n + \frac{\varepsilon n \log_{p_a} 2}{|\Sigma|}] = 1.$$

Note that the random variables  $X_a^n, X_b^n, \dots$  are independent for each  $a, b, \dots \in \Sigma$ , we have

$$\lim_{n \rightarrow \infty} \sum_{s \in T_\varepsilon^n} p_s = 1.$$

□

**Lemma 1.2.** Let  $\rho \in D(\mathcal{H}_A)$  be composed of quantum state ensemble  $\{\rho_a\}_{a \in \Sigma}$  with probability distribution  $p = (p_a)$  such that  $\rho = \sum_{a \in \Sigma} p_a \rho_a$ . For each  $t = a_1 \cdots a_n \in \Sigma^n$ , if we denote  $p_t = p_{a_1} \cdots p_{a_n}$ ,  $\rho_t = \rho_{a_1} \otimes \rho_{a_2} \otimes \cdots \otimes \rho_{a_n}$ , and  $\rho_{T_\varepsilon^n} = \sum_{s \in T_\varepsilon^n} p_s \rho_s$ , then

$$\lim_{n \rightarrow \infty} \|\rho^{\otimes n} - \rho_{T_\varepsilon^n}\|_1 = 0.$$

*Proof.* Note that the quantum state  $\rho^{\otimes n}$  can be decomposed into

$$\rho^{\otimes n} = \sum_{t \in \Sigma^n} \prod_{a \in \Sigma} p_a^{n_{a,t}} \rho_t.$$

If we denote

$$\rho_{T_\varepsilon^n} = \sum_{s \in T_\varepsilon^n} \prod_{a \in \Sigma} p_a^{n_{a,s}} \rho_s,$$

then

$$\lim_{n \rightarrow \infty} \sum_{t \in \Sigma^n \setminus T_\varepsilon^n} \prod_{a \in \Sigma} p_a^{n_{a,t}} = 0.$$

Therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\rho^{\otimes n} - \rho_{T_\varepsilon^n}\|_1 &= \lim_{n \rightarrow \infty} \left\| \sum_{t \in \Sigma^n \setminus T_\varepsilon^n} \prod_{a \in \Sigma} p_a^{n_{a,t}} \rho_t \right\| \\ &\leq \lim_{n \rightarrow \infty} \sum_{t \in \Sigma^n \setminus T_\varepsilon^n} \prod_{a \in \Sigma} p_a^{n_{a,t}} = 0. \end{aligned}$$

□

The *fidelity* between two quantum states  $\rho$  and  $\sigma$  is defined by

$$F(\rho, \sigma) = \text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}.$$

Let  $|\phi\rangle\langle\phi| \in \mathcal{D}(\mathcal{H}_A)$  be a pure state and  $\rho = \sum_{a \in \Sigma} p_a |\psi_a\rangle\langle\psi_a| \in \mathcal{D}(\mathcal{H}_A)$  be any quantum state. Note that for each  $\lambda \geq 0$ ,  $\sqrt{\lambda}|\phi\rangle\langle\phi| = \sqrt{\lambda}|\phi\rangle\langle\phi|$ , so

$$(F(|\phi\rangle\langle\phi|, \rho))^2 = \langle|\phi\rangle\langle\phi|, \rho\rangle = \sum_{a \in \Sigma} p_a \langle|\phi\rangle\langle\phi|, |\psi_a\rangle\langle\psi_a|\rangle = \sum_{a \in \Sigma} p_a (F(|\phi\rangle\langle\phi|, |\psi_a\rangle\langle\psi_a|))^2,$$

in particular,  $F(|\phi\rangle\langle\phi|, |\psi\rangle\langle\psi|) = |\langle\phi, \psi\rangle|$  for any unit vectors  $|\phi\rangle, |\psi\rangle \in \mathcal{H}_A$ .

Let  $\rho \in \mathcal{D}(\mathcal{H}_A)$ . A *purification* of  $\rho$  in  $\mathcal{H}_A \otimes \mathcal{H}_B$  is any pure state  $|u\rangle\langle u|$  in  $\mathcal{H}_A \otimes \mathcal{H}_B$  for which the partial trace  $\text{Tr}_{\mathcal{H}_B}(|u\rangle\langle u|)$  of  $|u\rangle\langle u|$  over subsystem  $\mathcal{H}_B$  is  $\rho$ .

Let  $\mathcal{T}(\mathcal{H}_A, \mathcal{K}_A)$  denote the set of all *linear super-operators* from  $\mathcal{L}(\mathcal{H}_A)$  to  $\mathcal{L}(\mathcal{K}_A)$ . We say that  $\Phi \in \mathcal{T}(\mathcal{H}_A, \mathcal{K}_A)$  is *completely positive* if for each positive integer  $k \in \mathbf{N}$ ,  $\Phi \otimes M_k : \mathcal{L}(\mathcal{H}) \otimes M_k \rightarrow \mathcal{L}(\mathcal{H}) \otimes M_k$  is positive, where  $M_k$  is the set of all  $k \times k$  complex matrices. It follows from the famous theorems of Choi [3] and Kraus [4] that if  $\Phi$  is complete positive, then it can be represented in the following form

$$\Phi(X) = \sum_{j=1}^n M_j X M_j^\dagger, \quad X \in \mathcal{L}(\mathcal{H}_A),$$

where  $\{M_j\}_{j=1}^n \subseteq \mathcal{L}(\mathcal{H}_A, \mathcal{K}_A)$ ,  $M^\dagger$  is the *adjoint operator* of  $M$ . In this case, we denote  $\Phi = \sum_j \text{Ad}_{M_j}$ . If  $\sum_\mu M_\mu^\dagger M_\mu = I_{\mathcal{H}_A}$ , then  $\Phi = \sum_j \text{Ad}_{M_j}$  is said to be an *admissible quantum operation*.

Let  $\Phi_A \in \mathcal{T}(\mathcal{H}_A, \mathcal{K}_A)$ ,  $\Phi_B \in \mathcal{T}(\mathcal{H}_B, \mathcal{K}_B)$  be two admissible quantum operations. Then  $\Phi_A \otimes \Phi_B \in \mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_B, \mathcal{K}_A \otimes \mathcal{K}_B)$  is said to be an *admissible product quantum operation*.

Let  $\mathcal{Z}_A$  be a finite dimensional complex Hilbert space,  $\Sigma$  be a finite set,

$$\{P_a : a \in \Sigma\}$$

be a measurement on  $\mathcal{Z}_A$ , and let  $\mathcal{Z}_B = \mathbf{C}^\Sigma$ ,  $\{e_a : a \in \Sigma\}$  be the standard orthogonal basis of  $\mathcal{Z}_B$ ,  $E_{a,a} = |e_a\rangle\langle e_a|$ . The super-operator of the form

$$\Phi \in \mathcal{T}(\mathcal{H}_A \otimes \mathcal{Z}_A \otimes \mathcal{H}_B, \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{Z}_B)$$

defined by

$$\Phi(X) = \sum_{a \in \Sigma} \text{Tr}_{\mathcal{Z}_A}[(I_{\mathcal{H}_A} \otimes P_a \otimes I_{\mathcal{H}_B})X] \otimes E_{a,a}$$

is an *Alice-Bob measurement transmission super-operator*.

The interpretation of such a super-operator is that Alice performs the measurement described by  $\{P_a : a \in \Sigma\}$  on the part of her quantum system  $\mathcal{Z}_A$  and transmits the result to Bob. We then

imagine that Bob initializes a quantum register to the state  $E_{a,a}$  for whichever outcome  $a \in \Sigma$  Alice obtained, so that we may incorporate this measurement outcome into the description of Bob's quantum information.

A *Bob-to-Alice measurement transmission super-operator* is a super-operator of the form

$$\Phi \in T(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{Z}_B, \mathcal{H}_A \otimes \mathcal{Z}_A \otimes \mathcal{H}_B)$$

that is defined in the same way as an Alice-Bob measurement transmission super-operator, except that of course Bob performs the measurement rather than Alice.

We may speak of a *measurement transmission super-operator* to mean either an Alice-to-Bob or Bob-to-Alice measurement transmission super-operator.

An LOCC super-operator is any super-operator of the form

$$\Phi \in T(\mathcal{H}_A \otimes \mathcal{H}_B, \mathcal{K}_A \otimes \mathcal{K}_B)$$

that can be obtained from the composition of any finite number of admissible product super-operators and measurement transmission super-operators.

We will write

$$\text{LOCC}(\mathcal{H}_A, \mathcal{K}_A : \mathcal{H}_B, \mathcal{K}_B)$$

to denote the collection of all LOCC super-operators as just defined. For much more notations and definitions, see [1].

Given a quantum state  $\rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ , denote  $\rho_A = \text{Tr}_{\mathcal{H}_B} \rho$ , and  $\rho_B = \text{Tr}_{\mathcal{H}_A} \rho$ , respectively. If  $\rho$  is the pure state  $|\psi\rangle\langle\psi|$  in  $D(\mathcal{H}_A \otimes \mathcal{H}_B)$ , then  $S(\rho_A) = S(\rho_B)$  [5]. The entanglement  $E(|\psi\rangle)$  of pure state  $|\psi\rangle\langle\psi|$  is defined by

$$E(|\psi\rangle) = S(\rho_A) = S(\rho_B).$$

Given a quantum state  $\rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ , consider all possible pure state ensemble  $\{|\psi_a\rangle\langle\psi_a|\}_{a \in \Sigma}$  with probability distribution  $p = (p_a)$  such that  $\rho = \sum_{a \in \Sigma} p_a |\psi_a\rangle\langle\psi_a|$ . The *entanglement of formation*  $E_f(\rho)$  of  $\rho$  is defined by ([5], [6]):

$$E_f(\rho) = \min_{\Sigma} \sum_{a \in \Sigma} p_a E(|\psi_a\rangle).$$

Let  $\tau \in D(\mathcal{C}^{\{0,1\}} \otimes \mathcal{C}^{\{0,1\}})$  denote the operator

$$\tau = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|).$$

The *entanglement cost*  $E_c(\rho)$  of  $\rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$  is the infimum over all real numbers  $\alpha \geq 0$  ([2], [5]), for which there exists a sequence of LOCC super-operators

$$\Phi_n \in LOCC((\mathcal{C}^{\{0,1\}})^{\otimes \lfloor \alpha n \rfloor}, \mathcal{H}_A^{\otimes n} : (\mathcal{C}^{\{0,1\}})^{\otimes \lfloor \alpha n \rfloor}, \mathcal{H}_B^{\otimes n}),$$

such that  $\lim_{n \rightarrow \infty} F(\Phi_n(\tau^{\otimes \lfloor \alpha n \rfloor}), \rho^{\otimes n}) = 1$ .

The *distillable entanglement*  $E_d(\rho)$  of a quantum state  $\rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$  is the supremum over all real numbers  $\alpha \geq 0$  ([5], [7]), for which there exists a sequence of LOCC super-operators

$$\Phi_n \in LOCC(\mathcal{H}_A^{\otimes n}, (\mathcal{C}^{\{0,1\}})^{\otimes \lfloor \alpha n \rfloor} : \mathcal{H}_B^{\otimes n}, (\mathcal{C}^{\{0,1\}})^{\otimes \lfloor \alpha n \rfloor}),$$

such that  $\lim_{n \rightarrow \infty} F(\Phi_n(\rho^{\otimes n}), \tau^{\otimes \lfloor \alpha n \rfloor}) = 1$ .

For each  $\rho, \sigma \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ , we have the following important facts ([2], [8], [9]):

$$(1). E_f(\rho \otimes \sigma) \leq E_f(\rho) + E_f(\sigma).$$

(2). For any family of CP local maps  $\mathcal{M}_{loc}^{(i)}(\rho) = \sum_{j,k} A_{ij} \otimes B_{ik} \rho A_{ij}^\dagger \otimes B_{ik}^\dagger$  such that  $\sum_{i,j,k} A_{ij}^\dagger A_{ij} \otimes B_{ik}^\dagger B_{ik} = 1$ . Then  $E_f(\rho)$  satisfies the monotonicity condition

$$\sum_i p_i E_f(p_i^{-1} \mathcal{M}_{loc}^{(i)}(\rho)) \leq E_f(\rho), \text{ with } p_i = \text{Tr}[\mathcal{M}_{loc}^{(i)}(\rho)].$$

$$(3). E_d(\rho) \leq E_c(\rho) \leq E_f(\rho).$$

(4). If  $\rho$  is a pure state  $|\psi\rangle\langle\psi|$ , then

$$E_f(|\psi\rangle\langle\psi|) = E_c(|\psi\rangle\langle\psi|) = E_d(|\psi\rangle\langle\psi|).$$

$$(5). E_c(\rho) = \lim_{n \rightarrow \infty} \frac{E_f(\rho^{\otimes n})}{n}.$$

$$(6). E_c(\rho^{\otimes k}) = k E_c(\rho), \text{ where } k = 1, 2, \dots$$

Generally, by using a similar approach to proof of fact (6), we can also have that

$$(7). E_d(\rho^{\otimes k}) = k E_d(\rho), k = 1, 2, \dots$$

(8). If  $\rho$  is a product of two pure state  $|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|$ , then

$$E_c(|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|) = E_d(|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|) = E(|\psi\rangle) + E(|\phi\rangle).$$

However, when  $\rho$  and  $\sigma$  are not the pure states, we do not know whether the additivity of  $E_c$  and  $E_d$  hold.

In [10], Professors P. Horodecki, R. Horodecki and M. Horodecki introduced the following important notion:

**Definition 1.3.** Let  $\mathcal{H} = \bigotimes_{l=1}^m \mathcal{H}_l$ . We say that two pure states  $|\psi\rangle\langle\psi|$  and  $|\phi\rangle\langle\phi|$  on  $\mathcal{H}$  are *k-locally orthogonal*, if there exist some  $k$  subsystems  $\mathcal{H}_{i_1}, \dots, \mathcal{H}_{i_k}$  such that

$$\text{Tr}((|\psi\rangle\langle\psi|)_l(|\phi\rangle\langle\phi|)_l) = 0, \quad l = i_1, \dots, i_k,$$

where  $(|\psi\rangle\langle\psi|)_l = \text{Tr}_{\bigotimes_{j=1, j \neq l}^m \mathcal{H}_j} |\psi\rangle\langle\psi|$ .

Moreover, the set of pure states  $\{|\psi_i\rangle\langle\psi_i|\}_{i=1}^K$  is said to be *locally orthogonal* if  $\{|\psi_i\rangle\langle\psi_i|\}_{i=1}^K$  can be ordered in the sequence  $\{|\psi_{i_l}\rangle\langle\psi_{i_l}|\}_{i_l=i_1}^{i_K}$  such that for any  $1 \leq l \leq K$ , the pure state  $|\psi_{i_l}\rangle\langle\psi_{i_l}|$  and the pure state  $|\psi_{i_n}\rangle\langle\psi_{i_n}|$  are *1-locally orthogonal* on the same subsystem whenever  $n > l$ .

They proved also the following conclusion [10]:

**Proposition 1.4.** Let  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ ,  $\rho$  be a quantum state on  $\mathcal{H}$ . If  $\rho$  is composed of the locally orthogonal pure state ensemble  $\{|\psi_i\rangle\langle\psi_i|\}_{i=1}^K$  with probability distribution  $p = (p_i)$  such that  $\rho = \sum_{i=1}^K p_i |\psi_i\rangle\langle\psi_i|$ , then

$$(i) \quad E_f(\rho) = \sum_i p_i E_f(|\psi_i\rangle\langle\psi_i|),$$

$$(ii) \quad E_d(\rho) = E_f(\rho) = E_c(\rho).$$

Proposition 1.4 showed that the locally orthogonal pure states can be distinguished by LOCC super-operators without destroying them.

## 2 The Local orthogonality between mixed states

In order to state our results, firstly, we need to extend the local orthogonality to the general quantum states.

**Definition 2.1.** Let  $\mathcal{H} = \bigotimes_{l=1}^m \mathcal{H}_l$ . We say that two quantum states  $\rho$  and  $\sigma$  on  $\mathcal{H}$  are *k-locally orthogonal*, if there exist some  $k$  subsystems  $\mathcal{H}_{i_1}, \dots, \mathcal{H}_{i_k}$ , such that

$$\text{Tr}((\rho)_l(\sigma)_l) = 0, \quad l = i_1, \dots, i_k,$$

where  $(\rho)_l = \text{Tr}_{\bigotimes_{j=1, j \neq l}^m \mathcal{H}_j} \rho$ .

Moreover, the set of quantum states  $\{\rho^{(a)}\}_{a \in \Sigma}$  on  $\mathcal{H}$  is said to be *locally orthogonal*, if  $\{\rho^{(a)}\}_{a \in \Sigma}$  can be ordered in the sequence  $\{\rho^{(a_1)}, \rho^{(a_2)}, \dots, \rho^{(a_K)}\}$  such that for each  $1 \leq q \leq K$ , the quantum state  $\rho^{(a_q)}$  and the quantum state  $\rho^{(a_n)}$  is *1-locally orthogonal* on the same subsystem whenever  $n > q$ , where  $|\Sigma| = K$ .

In order to prove our main results, we need also the following Proposition, which can be proved easily by the definition of local orthogonality:

**Proposition 2.2.** *Let  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ ,  $\rho_1$  and  $\rho_2$  are two locally orthogonal quantum states on  $\mathcal{H}$ . Then  $\langle \text{Tr}_A(\rho_1), \text{Tr}_A(\rho_2) \rangle = 0$  or  $\langle \text{Tr}_B(\rho_1), \text{Tr}_B(\rho_2) \rangle = 0$ . Moreover, if  $\mathcal{H}_A = \bigotimes_{i=1}^m \mathcal{H}_i^{(A)}$ ,  $\mathcal{H}_B = \bigotimes_{j=1}^n \mathcal{H}_j^{(B)}$ ,  $\langle \text{Tr}_A(\rho_1), \text{Tr}_A(\rho_2) \rangle = 0$ , then  $\text{Tr}_A(\rho_1)$  and  $\text{Tr}_A(\rho_2)$  are locally orthogonal on  $\mathcal{H}_B$ . If  $\langle \text{Tr}_B(\rho_1), \text{Tr}_B(\rho_2) \rangle = 0$ , then  $\text{Tr}_B(\rho_1)$  and  $\text{Tr}_B(\rho_2)$  are locally orthogonal on  $\mathcal{H}_A$ , too.*

### 3 Main results

In this section, we show that if the quantum state  $\rho$  is composed of the locally orthogonal quantum state ensemble  $\{\rho_a\}_{a \in \Sigma}$  with probability distribution  $p = (p_a)$  such that  $\rho = \sum_{a \in \Sigma} p_a \rho_a$ , then the entanglement of  $\rho$  can be decomposed into the entanglement of  $\{\rho_i\}$  without losing them.

**Lemma 3.1.** *Let  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ ,  $\rho, \sigma$  be quantum states on  $\mathcal{H}$ . If the entanglement cost of quantum product state  $\rho \otimes \sigma$  is additive, that is  $E_c(\rho \otimes \sigma) = E_c(\rho) + E_c(\sigma)$ , then*

$$\lim_{n_1, n_2 \rightarrow \infty} \frac{E_f(\rho^{\otimes n_1}) + E_f(\sigma^{\otimes n_2})}{n_1 + n_2} = \lim_{n_1, n_2 \rightarrow \infty} \frac{E_f(\rho^{\otimes n_1} \otimes \sigma^{\otimes n_2})}{n_1 + n_2},$$

where  $n_1, n_2 \in \mathbb{N}$ , and there exists a positive number  $p > 1$  such that  $\lim_{n_1, n_2 \rightarrow \infty} \frac{n_2}{n_1} = p$ .

*Proof.* Consider the additive of entanglement cost, by the property (5) of the entanglement, we have

$$\lim_{n \rightarrow \infty} \frac{E_f(\rho^{\otimes n}) + E_f(\sigma^{\otimes n})}{n} = \lim_{n \rightarrow \infty} \frac{E_f((\rho \otimes \sigma)^{\otimes n})}{n} < \infty.$$

Therefore, for each  $\varepsilon_1 > 0$ , there exists a number  $N_1 \in \mathbb{N}$ , such that for any  $n > N_1$ , we have

$$0 < \frac{E_f(\rho^{\otimes n}) + E_f(\sigma^{\otimes n}) - E_f((\rho \otimes \sigma)^{\otimes n})}{n} < \varepsilon_1,$$

and for each  $\varepsilon_2 > 0$ , there exists a number  $N_2 \in \mathbb{N}$ , such that for any  $n_2 > n_1 > N_1$ , we have

$$\left| \frac{E_f(\rho^{\otimes n_1})}{n_1} - \frac{E_f(\rho^{\otimes n_2})}{n_2} \right| < \varepsilon_2.$$

Now, let us assume that there exists a positive number  $a > 0$  such that

$$\lim_{n_1, n_2 \rightarrow \infty} \frac{E_f(\rho^{\otimes n_1}) + E_f(\sigma^{\otimes n_2}) - E_f(\rho^{\otimes n_1} \otimes \sigma^{\otimes n_2})}{n_1 + n_2} = a,$$

where  $n_1, n_2 \in \mathbb{N}$ , and there exists a positive number  $p > 1$  such that  $\lim_{n_1, n_2 \rightarrow \infty} \frac{n_2}{n_1} = p$  ( $n_1 < n_2$ ), then for each  $\varepsilon_3 > 0$ , there exists a number  $N_3 \in \mathbb{N}$ , such that for any  $n_2 > n_1 > N_1$ , we have

$$a - \varepsilon_3 < \frac{E_f(\rho^{\otimes n_1}) + E_f(\sigma^{\otimes n_2}) - E_f(\rho^{\otimes n_1} \otimes \sigma^{\otimes n_2})}{n_1 + n_2} < a + \varepsilon_3,$$



and for each  $\varepsilon_4 > 0$ , there exists a number  $N_4 \in \mathbb{N}$ , such that for any  $n_2 > n_1 > N_4$ , we have

$$p - \varepsilon_4 < \frac{n_2}{n_1} < p + \varepsilon_4.$$

Therefore, when  $N = \max\{N_1, N_2, N_3, N_4\}$ , for each  $n_2 > \min\{n_1, n_2 - n_1\} > N$ , it follows that

$$\begin{aligned} -(a + \varepsilon_3) &< \frac{E_f(\rho^{\otimes n_2}) + E_f(\sigma^{\otimes n_2}) - E_f((\rho \otimes \sigma)^{\otimes n_2})}{n_1 + n_2} - \frac{E_f(\rho^{\otimes n_1}) + E_f(\sigma^{\otimes n_2}) - E_f(\rho^{\otimes n_1} \otimes \sigma^{\otimes n_2})}{n_1 + n_2} \\ &= \frac{E_f(\rho^{\otimes n_2}) - E_f(\rho^{\otimes n_1}) - E_f((\rho \otimes \sigma)^{\otimes n_2}) + E_f(\rho^{\otimes n_1} \otimes \sigma^{\otimes n_2})}{n_1 + n_2} \\ &< -(a - \frac{(p + \varepsilon_4)\varepsilon_1}{1 + p - \varepsilon_4} - \varepsilon_3), \end{aligned}$$

note that

$$\frac{E_f(\rho^{\otimes(n_2-n_1)})}{n_1 + n_2} - \varepsilon_2 < \frac{E_f(\rho^{\otimes n_2}) - E_f(\rho^{\otimes n_1})}{n_1 + n_2} < \frac{E_f(\rho^{\otimes(n_2-n_1)})}{n_1 + n_2} + \varepsilon_2,$$

then we obtain that

$$\begin{aligned} -(a + \varepsilon_2 + \varepsilon_3) &< \frac{E_f(\rho^{\otimes(n_2-n_1)}) - E_f((\rho \otimes \sigma)^{\otimes n_2}) + E_f(\rho^{\otimes n_1} \otimes \sigma^{\otimes n_2})}{n_1 + n_2} \\ &< -(a - \frac{(p + \varepsilon_4)\varepsilon_1}{1 + p - \varepsilon_4} - \varepsilon_2 - \varepsilon_3). \end{aligned}$$

But we choose  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 > 0$  small enough so that  $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 < a/3$  and  $\varepsilon_4 < 1/2$ , then

$$E_f(\rho^{\otimes(n_2-n_1)}) + E_f(\rho^{\otimes n_1} \otimes \sigma^{\otimes n_2}) < E_f(\rho^{\otimes(n_2-n_1)} \otimes \rho^{\otimes n_1} \otimes \sigma^{\otimes n_2}) = E_f((\rho \otimes \sigma)^{\otimes n_2}),$$

this contradicts the property (1) of entanglement. Hence, we have

$$\lim_{n \rightarrow \infty} \frac{E_f(\rho^{\otimes n_1}) + E_f(\sigma^{\otimes n_2}) - E_f(\rho^{\otimes n_1} \otimes \sigma^{\otimes n_2})}{n_1 + n_2} = 0,$$

which completes the proof.  $\square$

**Theorem 3.2.** Let  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ ,  $\rho$  be a quantum state on  $\mathcal{H}$ . If  $\rho$  is composed of the locally orthogonal quantum ensemble  $\{\rho_a\}_{a \in \Sigma}$  with probability distribution  $p = (p_a)$  such that  $\rho = \sum_{a \in \Sigma} p_a \rho_a$ , then

$$E_f(\rho) = \sum_{a \in \Sigma} p_a E_f(\rho_a).$$

Moreover, if the entanglement cost  $E_c$  is additive for the quantum product state  $\otimes_{a \in \Sigma} \rho_a$ , that is,  $E_c(\otimes_{a \in \Sigma} \rho_a) = \sum_{a \in \Sigma} E_c(\rho_a)$ , then

$$E_c(\rho) = \sum_{a \in \Sigma} p_a E_c(\rho_a).$$

*Proof.* If  $\rho = p_1\rho_1 + p_2\rho_2$ , where  $p = (p_1, p_2)$  is a probability distribution, the quantum states  $\rho_1, \rho_2$  are locally orthogonal. By the definition of locally orthogonal, we know that

$$\langle \text{Tr}_A(\rho_1), \text{Tr}_A(\rho_2) \rangle = 0, \text{ or } \langle \text{Tr}_B(\rho_1), \text{Tr}_B(\rho_2) \rangle = 0.$$

Without lost generality, we assume that  $\langle \text{Tr}_A(\rho_1), \text{Tr}_A(\rho_2) \rangle = 0$ , then by Proposition 2.2,  $\text{Tr}_A(\rho_1)$  and  $\text{Tr}_A(\rho_2)$  are orthogonal, it implies that there are subspaces  $\mathcal{V}_i^B \subseteq \mathcal{H}_B$  such that  $\rho_i \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{V}_i^B)$  and  $\mathcal{V}_2^B \subseteq (\mathcal{V}_1^B)^\perp$ . The inequality  $E_f(\rho) \leq p_1 E_f(\rho_1) + p_2 E_f(\rho_2)$  follows from convexity of  $E_f$ . The reverse inequality is a consequence of the monotonicity property (2) of the entanglement applied to the maps

$$\mathcal{M}_{loc}^{(i)}(\rho) = 1^A \otimes \pi_i^B \rho 1^A \otimes \pi_i^B, \quad i = 1, 2, 3,$$

where  $\mathcal{V}_3^B = 1_B - (\mathcal{V}_1^B + \mathcal{V}_2^B)$ , and  $\pi_i^B$  are the projectors onto  $\mathcal{V}_i^B$ , respectively. It follows that

$$\sum_{a=1,2} p_a E_f(\rho_a) = \sum_{i=1}^3 q_i E_f(q_i^{-1} \mathcal{M}_{loc}^{(i)}(\rho)) \leq E_f(\rho),$$

with  $q_i = \text{Tr}[\mathcal{M}_{loc}^{(i)}(\rho)]$ .

Repeatedly, when the set of quantum states  $\{\rho_a\}$  is locally orthogonal, then

$$E_f(\rho) = \sum_a p_a E_f(\rho_a).$$

Next, regarding the entanglement cost  $E_c(\rho)$ , let the quantum state  $\rho^{\otimes n} = \sum_{t \in \Sigma^n} p_t \rho_t$  such that all quantum states  $\rho_t = \rho_{a_1} \otimes \rho_{a_2} \otimes \cdots \otimes \rho_{a_n}$ , where  $t = a_1 \cdots a_n$ . Then, by property of locally orthogonal quantum states  $\{\rho_t\}$ , we have

$$E_f(\rho^{\otimes n}) = \sum_{t \in \Sigma^n} p_t E_f(\rho_t).$$

And, for any positive real number  $\varepsilon$ , by the property (5) of the entanglement and Lemma 1.2, we have

$$\begin{aligned} E_c(\rho) &= \lim_{n \rightarrow \infty} \frac{E_f(\rho^{\otimes n})}{n} = \lim_{n \rightarrow \infty} \frac{\sum_{t \in \Sigma^n} p_t E_f(\rho_t)}{n} \\ &= \lim_{n \rightarrow \infty} \left[ \sum_{s \in T_\varepsilon^n} \left\{ \left( \prod_{a \in \Sigma} p_a^{n_{a,s}} \right) \frac{E_f(\rho_s)}{n} \right\} \right], \end{aligned}$$

where  $\sum_{a \in \Sigma} n_{a,s} = n$  for  $s \in T_\varepsilon^n$ . Also, by Lemma 3.1, if we have  $E_c(\bigotimes_{a \in \Sigma} \rho_a) = \sum_{a \in \Sigma} E_c(\rho_a)$ , then

$$\lim_{n \rightarrow \infty} \frac{E_f(\rho_s)}{n} = \lim_{n \rightarrow \infty} \frac{\sum_{a \in \Sigma} E_f(\rho_a^{\otimes n_{a,s}})}{n}.$$

Therefore

$$\begin{aligned}
E_c(\rho) &= \lim_{n \rightarrow \infty} \left[ \sum_{s \in T_\varepsilon^n} \left\{ \left( \prod_{a \in \Sigma} p_a^{n_{a,s}} \right) \sum_{a \in \Sigma} \frac{(n_{a,s}) E_f(\rho_a^{\otimes n_{a,s}})}{n(n_{a,s})} \right\} \right] \\
&= \lim_{n \rightarrow \infty} \left[ \sum_{s \in T_\varepsilon^n} \left\{ \left( \prod_{a \in \Sigma} p_a^{n_{a,s}} \right) \sum_{a \in \Sigma} \frac{(p_a) E_f(\rho_a^{\otimes n_{a,s}})}{(n_{a,s})} \right\} \right] \\
&\quad + \lim_{n \rightarrow \infty} \left[ \sum_{s \in T_\varepsilon^n} \left\{ \left( \prod_{a \in \Sigma} p_a^{n_{a,s}} \right) \sum_{a \in \Sigma} \frac{\{(n_{a,s}/n) - p_a\} E_f(\rho_a^{\otimes n_{a,s}})}{(n_{a,s})} \right\} \right].
\end{aligned}$$

It follows from  $|\frac{n_{a,s}}{n} - p_a| \leq \frac{\varepsilon \log p_a 2}{|\Sigma|} \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and  $n \rightarrow \infty$  that

$$\lim_{n \rightarrow \infty} \left[ \sum_{s \in T_\varepsilon^n} \left\{ \left( \prod_{a \in \Sigma} p_a^{n_{a,s}} \right) \sum_{a \in \Sigma} \frac{\{(n_{a,s}/n) - p_a\} E_f(\rho_a^{\otimes n_{a,s}})}{(n_{a,s})} \right\} \right] = 0,$$

and, by  $\sum_{s \in T_\varepsilon^n} (\prod_{a \in \Sigma} p_a^{n_{a,s}}) \rightarrow 1$  as  $n \rightarrow \infty$ , we have

$$\lim_{n \rightarrow \infty} \left[ \sum_{s \in T_\varepsilon^n} \left\{ \left( \prod_{a \in \Sigma} p_a^{n_{a,s}} \right) \sum_{a \in \Sigma} \frac{(p_a) E_f(\rho_a^{\otimes n_{a,s}})}{(n_{a,s})} \right\} \right] = \sum_{a \in \Sigma} p_a E_c(\rho_a).$$

□

For the distillable entanglement, we also hope to have the same decomposition under the condition of locally orthogonal. But it is hard to find certain conditions for holding the same result as equality, we only have a weak result as inequality.

Firstly, we need the following lemma for our result.

**Lemma 3.3.** *Consider any tensor product quantum states  $\rho \otimes \varrho$  of the quantum system composed from two subsystems, then*

$$E_d(\rho \otimes \varrho) \geq E_d(\rho) + E_d(\varrho).$$

*Proof.* Assume that  $E_d(\rho) = \alpha$ ,  $E_d(\sigma) = \beta$ , then there exists sequences  $\{\Phi_n, \Psi_n : n \in \mathbb{N}\}$  of LOCC super-operators such that

$$\lim_{n \rightarrow \infty} F(\Phi_n(\rho^{\otimes n}), \tau^{\otimes \lfloor \alpha n \rfloor}) = 1, \quad \lim_{n \rightarrow \infty} F(\Psi_n(\sigma^{\otimes n}), \tau^{\otimes \lfloor \beta n \rfloor}) = 1,$$

where  $\tau = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$ . Therefore, we can construct a sequence  $\{\Xi_n : n \in \mathbb{N}\}$  of LOCC super-operators such that

$$\lim_{n \rightarrow \infty} F(\Xi_n((\rho \otimes \sigma)^{\otimes n}), \tau^{\otimes (\lfloor \alpha n \rfloor + \lfloor \beta n \rfloor)}) = 1.$$

Firstly, if at least one of the two numbers  $\alpha, \beta$  are integers, note that

$$\lfloor \alpha n \rfloor + \lfloor \beta n \rfloor = \lfloor (\alpha + \beta)n \rfloor,$$

and  $E_d(\rho \otimes \sigma)$  is the supremum, we have  $E_d(\rho \otimes \sigma) \geq E_d(\rho) + E_d(\sigma)$  as required.

Next, if all numbers  $\alpha, \beta$  are not integers, then for  $\varepsilon > 0$  small enough so that  $0 < \alpha + \beta - \varepsilon$ , there exists an integer  $N$  such that for all  $n \geq N$ ,

$$\lfloor (\alpha + \beta - \varepsilon)n \rfloor \leq \lfloor \alpha n \rfloor + \lfloor \beta n \rfloor \leq \lfloor (\alpha + \beta)n \rfloor.$$

Also, for all  $n$ , we know  $\text{Tr}_{(\mathcal{C}^{\{0,1\}} \otimes \mathcal{C}^{\{0,1\}})^{\otimes \gamma_n}}$  are LOCC super-operators, where  $\gamma_n = \lfloor \alpha n \rfloor + \lfloor \beta n \rfloor - \lfloor (\alpha + \beta - \varepsilon)n \rfloor$ , thus,

$$\lim_{n \rightarrow \infty} F(\text{Tr}_{(\mathcal{C}^{\{0,1\}} \otimes \mathcal{C}^{\{0,1\}})^{\otimes \gamma_n}} \Xi_n((\rho \otimes \sigma)^{\otimes n}), \tau^{\otimes (\lfloor (\alpha + \beta - \varepsilon)n \rfloor)}) = 1.$$

Therefore, by the arbitrariness of  $\varepsilon$ , it follows that  $E_d(\rho \otimes \sigma) \geq E_d(\rho) + E_d(\sigma)$ .  $\square$

**Theorem 3.4.** Let  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ ,  $\rho$  be a quantum state on  $\mathcal{H}$ . If  $\rho$  is composed of the locally orthogonal quantum ensemble  $\{\rho_a\}_{a \in \Sigma}$  with probability distribution  $p = (p_a)$  such that  $\rho = \sum_{a \in \Sigma} p_a \rho_a$ , then

$$E_d(\rho) \geq \sum_{a \in \Sigma} p_a E_d(\rho_a).$$

*Proof.* Consider the distillable entanglement of quantum state  $\rho = \sum_{a \in \Sigma} p_a \rho_a$ . By definition of the distillable entanglement and Lemma 1.2,  $E_d(\rho)$  is the supremum over all real numbers  $\alpha \geq 0$  for which there exists a sequence of LOCC super-operators  $\{\Phi_n : n \in \mathbb{N}\}$ , such that

$$\lim_{n \rightarrow \infty} F(\Phi_n(\rho^{\otimes n}), \tau^{\otimes \lfloor \alpha n \rfloor}) = \lim_{n \rightarrow \infty} F(\Phi_n(\rho_{T_\varepsilon^n}), \tau^{\otimes \lfloor \alpha n \rfloor}) = 1,$$

where  $\tau = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$ , and the set  $T_\varepsilon^n$  is the typical set for the probability distribution  $p$ . It follows from the observation that the state  $\tau^{\otimes n}$  is a pure state for all positive numbers  $n \geq 1$  that

$$\begin{aligned} \lim_{n \rightarrow \infty} (F(\Phi_n(\rho^{\otimes n}), \tau^{\otimes \lfloor \alpha n \rfloor}))^2 &= \lim_{n \rightarrow \infty} (F(\Phi_n(\rho_{T_\varepsilon^n}), \tau^{\otimes \lfloor \alpha n \rfloor}))^2 \\ &= \lim_{n \rightarrow \infty} \left[ \sum_{s \in T_\varepsilon^n} \left\{ \left( \prod_{a \in \Sigma} p_a^{n_{a,s}} \right) (F(\Phi_n(\rho_s), \tau^{\otimes \lfloor \alpha n \rfloor}))^2 \right\} \right]. \end{aligned}$$

Also, because the locally orthogonal set can be distinguished by using LOCC super-operators without destroying them [10], there exists a sequence of LOCC super-operators  $\{\Phi_n : n \in \mathbb{N}\}$  for all quantum states  $\rho_s$ , such that

$$\lim_{n \rightarrow \infty} F(\Phi_n(\rho_s), \tau^{\otimes \lfloor \beta n \rfloor}) = 1,$$

where  $\beta = \lim_{n \rightarrow \infty} \frac{E_d(\bigotimes_{a \in \Sigma} \rho_a^{\otimes n_{a,s}})}{n}$ , thus

$$\lim_{n \rightarrow \infty} F(\Phi_n(\rho^{\otimes n}), \tau^{\otimes \lfloor \beta n \rfloor}) = \lim_{n \rightarrow \infty} \sqrt{\sum_{s \in T_\xi^n} \{(\prod_{a \in \Sigma} p_a^{n_{a,s}})(F(\Phi_n(\rho_s), \tau^{\otimes \lfloor \beta n \rfloor}))^2\}} = 1.$$

Therefore, by Lemma 3.3, we have

$$E_d(\rho) \geq \lim_{n \rightarrow \infty} \frac{E_d(\bigotimes_{a \in \Sigma} \rho_a^{\otimes n_{a,s}})}{n} \geq \sum_a p_a E_d(\rho_a)$$

as required.  $\square$

## 4 Example of main result

Finally, we present an interesting example to show that the conditions of Theorem 3.2 and Theorem 3.4 are existence.

**Example 4.1.** Consider  $\mathcal{H}_A = \mathcal{C}^3$ ,  $\mathcal{H}_B = \mathcal{C}^6$ , and the subspace  $\mathcal{H}_V \subseteq \mathcal{H}_A \otimes \mathcal{H}_B$  spanned by

$$\begin{aligned} |0\rangle_V &\equiv \frac{1}{2}(|1\rangle_A |2\rangle_B + |2\rangle_A |1\rangle_B + \sqrt{2}|0\rangle_A |3\rangle_B), \\ |1\rangle_V &\equiv \frac{1}{2}(|2\rangle_A |0\rangle_B + |0\rangle_A |2\rangle_B + \sqrt{2}|1\rangle_A |4\rangle_B), \\ |2\rangle_V &\equiv \frac{1}{2}(|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B + \sqrt{2}|0\rangle_A |5\rangle_B). \end{aligned}$$

Then for each  $\rho_V \in D(\mathcal{H}_V)$  and  $\sigma \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ , it follows from [12] that  $E_f(\rho_V \otimes \sigma) = E_f(\rho_V) + E_f(\sigma)$ . This implies that

$$E_c(\rho_V \otimes \sigma) = E_c(\rho_V) + E_c(\sigma). \quad (4.1)$$

In this condition, let us consider the quantum states  $\rho_V = \sum_{a \in \Sigma_V} p_a |\phi_a\rangle_V \langle \phi_a|$ ,  $\sigma = \sum_{b \in \Sigma_{AB}} q_b |\psi_b\rangle_{AB} \langle \psi_b|$  for any probability distributions  $p = (p_a)$  and  $q = (q_b)$ , where

$$\begin{aligned} |\phi_a\rangle_V &= r_a |1\rangle_V + \sqrt{1 - r_a^2} |2\rangle_V \quad (r_a \in [-1, 1]), \\ |\psi_b\rangle_{AB} &= (s_{0,b} |0\rangle_A + s_{1,b} |1\rangle_A + s_{2,b} |2\rangle_A) |3\rangle_B, \quad \text{and} \quad s_{0,b}^2 + s_{1,b}^2 + s_{2,b}^2 = 1. \end{aligned}$$

Then we know the set of quantum states  $\{\rho_V, \sigma\}$  is locally orthogonal. Consequently, the conditions of Theorem 3.2 and Theorem 3.4 can be satisfied.

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